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Okubo Second-Order Mass Formula from $SO(6,1)$

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Abstract

The study of the unification $SO(6,1)$ as a spectrum generating algebra is continued to include baryons in the scheme. Among others, the second-order mass formula of Okubo for the $\frac{3}{2}^+$ baryon resonances is derived.

1. *Introduction*

In a previous paper (Tait, 1972), hereafter referred to as I, the algebra $SO(6,1)$ was considered as a unification (Flato & Sternheimer, 1966) of the internal symmetry algebra $SO(6) \approx SU(4)$ and $SO(4,1)$ which is isomorphic to the algebra of the group of motion of one of the De Sitter space-times; the latter being chosen as an alternative to the Poincaré algebra which encounters serious difficulties (O'Raifeartaigh, 1965a, b, c; Jost, 1966; Segal, 1967) because of the 'no-go' theorems on the spectrum of the mass-squared operator $p_\mu p^\mu$. Although there can be no logical objection to the use of the De Sitter algebra as a viable space-time symmetry, what is surprising is the accuracy of the mass formulae obtained for the meson resonances 1^- and 2^+, when the expectation values of the mass operator were evaluated in a simple representation of $SO(6,1)$. This is because the only input which can remotely be connected with dynamical information is the choice of algebra and the choice of representation.

In this paper, more general representations are considered so that baryons may be included in the scheme, and the Okubo (1963) second-order mass formula (for M^2) is derived. Also, new mass sum rules for the 1^- and 2^+ meson resonances are found.

2. *Bilocal Harmonics*

The representation of $SO(6,1)$ considered in I was realised on the spherical harmonics

$$\psi^{p,q}_{T,T_3,Y}(\theta, \xi, \phi_1, \phi_2, \phi_3)$$

of S_5, given by

$$\sum_{a=1}^{6} x_a{}^2 = \sum_{i=1}^{3} z_i z_i{}^* = 1$$

The angles parametrise S_5, p and q are the two positive integers specifying a representation of $SU(3)$, while T, T_3, and Y represent isospin, its third component and hypercharge. These functions which carry all the representations of $SU(3)$ are just the components of irreducible Cartesian tensors constructed out of a single point on the surface S_5, being completely symmetric, traceless, with p contravariant and q covariant indices. In terms of the single vector (z_1, z_2, z_3) and its complex conjugate, they are homogeneous polynomials of degree $n = p + q$. In addition to these properties they must be eigenfunctions of the invariants of the algebras in the chain

$$SO(6,1) \supset SU(4) \supset SU(3) \oplus Z \supset SU(2)_T \oplus Y \supset T_3$$

where Z is the charm operator with eigenvalues $\frac{1}{2}(p-q)$ in this representation.

Let z_i and $z_i{}^*$ carry the contravariant and covariant indices respectively in the monomials

$$(abc|def) = z_1{}^a z_2{}^b z_3{}^c z_1^{*d} z_2^{*e} z_3^{*f} \tag{2.1}$$

where a, b, c, d, e, f are all positive integers or zero and $p = a + b + c$, $q = d + e + f$. An irreducible representation (I.R.) of $SO(6,1)$ can be defined on the space (2.1) using the form of the generators $L_{AB}(A, B = 0, 1, \ldots 6)$ described in I and the fact that $z_1 = x_4 + ix_6$, $z_2 = x_3 + ix_5$, $z_3 = x_1 + ix_2$. The non-compact generators L_{0a} may be combined into

$$E_1{}^\pm = L_{04} \pm iL_{06}, \qquad E_2{}^\pm = L_{03} \pm iL_{05}, \qquad E_3{}^\pm = L_{01} \pm iL_{02}$$

given by

$$E_i{}^+ = -2i\frac{\partial}{\partial z_i{}^*} + iz_i \sum_{k=1}^{3} \left(z_k \frac{\partial}{\partial z_k} + z_k{}^* \frac{\partial}{\partial z_k{}^*}\right) + \tfrac{5}{2}iz_i$$

$$E_i{}^- = -2i\frac{\partial}{\partial z_i} + iz_i{}^* \sum_{k=1}^{3} \left(z_k \frac{\partial}{\partial z_k} + z_k{}^* \frac{\partial}{\partial z_k{}^*}\right) + \tfrac{5}{2}iz_i{}^*$$

All that is necessary to make this representation identical to the one constructed in I is to require that the algebra acts on the traceless projections of (2.1) and that appropriate linear combinations of the same degree are formed to be eigenfunctions of T^2. With the inner product defined by the measure on S_5,

$$d\mu = \sin^3\theta \cos\theta \sin\xi \cos\xi \, d\theta \, d\xi \prod_{k=1}^{3} d\phi_k$$

two monomials ψ and ψ' are orthogonal unless $T_3 = T_3'$, $Y = Y'$, $Z = Z'$; and the normalisation of $(abc|def)$ is given by

$$N^2 = 8\pi^3 \frac{[2(a+d)]!![2(b+e)]!![2(c+f)]!!}{[2(n+2)]!!}$$

Now consider functions of the components of two distinct vectors in $\mathbb{C}_3$, say z_i and u_i, together with their complex conjugates, where the locus of both vectors is still S_5. Let them be defined in the following way (Bég & Ruegg, 1965)

$$\psi_\sigma{}^n(u,z) = \sum_{n',n'',\nu} \sum_{\sigma',\sigma''} \psi_{\sigma'}^{n'}(z)\, \psi_{\sigma''}^{n''}(u) \begin{bmatrix} n' & n'' & n_\nu \\ \sigma' & \sigma'' & \sigma \end{bmatrix}$$

where the last symbol is an $SU(4)$ Clebsh–Gordon coefficient, the functions $\psi_\sigma{}^n$ are the normalised spherical harmonics found previously, corresponding to the representation $[0,n,0]$ of $SU(4)$, and σ denotes collectively the quantum numbers (p, T, T_3, Y). The $\psi_\sigma{}^n(u,z)$ are called bilocal harmonics, and if they are to provide an irreducible representation space for $SO(6,1)$, it is necessary to find the irreducible subspaces contained in the direct product

$$\left(\bigoplus_{m=0}^{\infty} [0,m,0]\right)_z \oplus \left(\bigoplus_{n=0}^{\infty} [0,n,0]\right)_u$$

where

$$[0,m,0] \otimes [0,n,0] = \bigoplus_{\substack{y,x,r \\ y+x+r=n}} [r, m-y+x-r, r]; \qquad (m \geqslant n) \quad (2.2)$$

By varying m and n in (2.2) it is easily seen that an I.R. of $SO(6,1)$ may be constructed in the space of all the real representations of $SU(4)$. The problem of finding the irreducible subspaces is not attempted here, however. But instead certain representations of $SU(4)$, containing representations of $SU(3)$ of physical interest (such as octets and decuplets), are selected and examined. These are the **15** and **20**$''$ containing the octets

$$\psi_{ij}^{\mp} = z_i u_j^* \mp u_i z_j^* - \tfrac{1}{3}\delta_{ij}(\mathbf{u}^\dagger \mathbf{z} \mp \mathbf{z}^\dagger \mathbf{u})$$

respectively, the **50** containing the decuplet

$$\psi_{ijk} = z_i z_j u_k + z_i z_k u_j + z_k z_j u_i$$

and finally the **64** containing the octet

$$\psi_i{}^j = z_i \varepsilon^{jkl} z_k u_l$$

In $\psi^{\pm}$, the complex conjugates of the basis functions also lie in the respective octets, so that they represent mesons. Since this is not the case for the other two multiplets, we can make a correspondence with baryons. It happens that baryons and antibaryons are contained in the same $SU(4)$ multiplet. The diagonal matrix elements of the mass-squared operator can now be examined with these functions using the fact that the inner product is now defined by

$$(\psi, \psi') = \iint \psi^* \psi' \, d\mu_z \, d\mu_u$$

where $d\mu_z(d\mu_u)$ is the surface element on S_5 traced out by the vector $\mathbf{z}(\mathbf{u})$

3. *Mass Formulae*

The mass-squared operator is

$$\begin{aligned} M^2 &= \mu^2\{L_{01}^2 + L_{02}^2 + L_{03}^2 + L_{05}^2 - L_{12}^2 - L_{23}^2 - L_{31}^2 - L_{15}^2 - L_{25}^2 - L_{35}^2\} \\ &= \mu^2\{\tfrac{1}{2}(E_2^+ E_2^- + E_2^- E_2^+ + E_3^+ E_3^- + E_3^- E_3^+) - 2C^2 - 2T^2\} \end{aligned}$$

where C^2 and the unknown parameter μ^2 are defined in I. The operators $L_{AB}(u, z)$ representing $SO(6,1)$ in the space of bilocal harmonics are given by

$$L_{AB}(u, z) = L_{AB}(z) \otimes I(u) + I(z) \otimes L_{AB}(u)$$

in an obvious notation; and the commutation relations are unaffected by the substitutions $E_1^+(z) \to E_i^+(z) + \gamma z_i$, $E_i^+(u) \to E_i^+(u) + \gamma_1 u_i$ (see I), while the representation remains hermitian only if γ and γ_1 are real. The computation of the matrix elements $m^2(T, Y)$ of M^2 yields the following results.

3.1. *The Decuplet* ψ_{ijk}

$$\begin{aligned} m^2(\tfrac{3}{2}, 1) &= \mu^2[\tfrac{591}{80} + \tfrac{4}{5}\gamma^2 + \tfrac{3}{4}\gamma_1{}^2] \\ m^2(1, 0) &= \mu^2[\tfrac{70}{6} + \tfrac{2}{3}\gamma^2 + \tfrac{2}{3}\gamma_1{}^2] \\ m^2(\tfrac{1}{2}, -1) &= \mu^2[\tfrac{3347}{240} + \tfrac{8}{15}\gamma^2 + \tfrac{7}{12}\gamma_1{}^2] \\ m^2(0, -2) &= \mu^2[\tfrac{569}{40} + \tfrac{2}{5}\gamma^2 + \tfrac{1}{2}\gamma_1{}^2] \end{aligned}$$

which leads to a consistency sum rule, accurately satisfied by the $3/2^+$ decuplet N^*, Y_1^*, Ξ^*, Ω^-:

$$\underset{(8{\cdot}550\ \mathrm{GeV}^2)}{N^{*2} + 3\Xi^{*2}} = \underset{(8{\cdot}550\ \mathrm{GeV}^2)}{\Omega^{-2} + 3\,Y_1^{*2}} \tag{3.1.1}$$

(3.1.1) can be derived from the equal spacing rule for linear masses, or from the Okubo (1963) second-order mass formula (for mass-squared), namely

$$M^2 = a + bY + cY^2$$

3.2. *The Octet* ψ_{ij}^+

$$\begin{aligned} m^2(1, 0) &= \mu^2[\tfrac{67}{8} + \tfrac{3}{4}(\gamma^2 + \gamma_1{}^2)] \\ m^2(\tfrac{1}{2}, 1) &= \mu^2[\tfrac{181}{16} + \tfrac{5}{8}(\gamma^2 + \gamma_1{}^2)] \\ m^2(0, 0) &= \mu^2[\tfrac{109}{8} + \tfrac{7}{12}(\gamma_2 + \gamma_1{}^2)] \end{aligned}$$

sum rule:

$$64m^2(\tfrac{1}{2}, 1) = 23m^2(1, 0) + 39m^2(0, 0)$$

which is well satisfied for the meson resonances 1^- (l.h.s. = 51·0, r.h.s. = 54·0) and 2^+ (l.h.s. = 128·6, r.h.s. = 128·4).

The significance of the other two octets is not so clear-cut. $\psi_{\bar{i}j}$ yields the sum rule $24m^2(\frac{1}{2},1) = 13m^2(1,0) + 9m^2(0,0)$, which is satisfied by no known meson octet. $\psi_i{}^j$ gives $13\Lambda^{*2} = 12\Xi^{*2} + 3\Sigma^{*2}$ (the minor of N^{*2} is zero). Very tentatively, we assign the $\frac{7}{2}^-$ octet of baryons to this multiplet; for $N^*(2190)$, Λ^* (2100) have been established as $\frac{7}{2}^-$, while Ξ^* (1815) lies in the sequence $\frac{3}{2}^-$, $\frac{5}{2}^+$, $\frac{7}{2}^-$ ... (Dalitz, 1969). Using these as input, they imply a Σ^* with mass 2439 MeV. This is taken to be the Σ^* (2455) which as yet has no spin-parity assignment.

The numerical calculations indicated above show that there is a need for a closer examination of the representations of $SO(6,1)$ and also for a study of other algebras containing $SU(3)$ and $SO(4,1)$.

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